Time limit: 20 minutes
Instructions: For this test, you work in your teams of three to solve 50 short answer questions.
No calculators.

1. How many cubic centimeters are in 5 cubic meters?
2. Find the sum of the first 20 positive integers.
3. Find the number of ways to rearrange the letters in $SPEED$ (Note that if the two Es switch, then the arrangement is considered the same).
4. Calculate $(\sqrt{17} - 2\sqrt{2})(\sqrt{17} + 2\sqrt{2})$.
5. Find the minimum area of a nondegenerate right triangle with integer side lengths.
6. The circumference of a circle is 10π. If the area is $x \cdot \pi$, find x.
7. Jonah drops a peanut on the floor. It lands at point A, bounces 20 centimeters to the right to point B, then 15 centimeters down to point C, and lastly, 12 centimeters left to point D. Find the distance in centimeters between point A and point D.
8. The triangle $\triangle ABC$ has side lengths $AB = BC = 41$ and $AC = 18$. What is the area of $\triangle ABC$?
9. Compute the units digit of $333^1 \cdot 333^2 \cdot 333^3 \cdot \cdots \cdot 333^{2019} \cdot 333^{2020}$.
10. How many perfect squares are there between 1000 and 10000, inclusive?
11. The volume of a certain sphere is 32 m^3. Consider a second sphere with radius half as large as the radius of the first sphere. Find the volume of the second sphere.
12. What is the sum of all positive integers whose cubes are less than 2020?
13. Let x be a two-digit integer. Let x' be the two-digit integer obtained by reversing the digits of x. For how many such values of x is the property $|x - x'| = 54$ satisfied?
14. Find the sum of all real values of x that satisfy the equation
$$\left(3x^2 + 2\right)^{\left(4x^2 - 1\right)} + 2^3 = 3^2.$$
15. Andy starts with a number, takes its square root, multiplies the result by -4, adds 2, squares the result, then subtracts 6. If the number he ends with is 30, then what number did he start with?
16. A cube is colored blue on four of its faces, and then cut into $7^3 = 343$ smaller cubes. If the probability that a randomly selected face of a randomly selected subcube is blue is $\frac{a}{b}$ where a and b are relatively prime positive integers, find $a + b$.

17. Define A_0 to be a square of side length 1. Define A_1 as the regular octagon obtained by cutting off a portion of each corner of A_0. Define A_2 as the regular 16-gon obtained by cutting off a portion of each corner of A_1. If this process goes on forever, then the perimeter of A_{2020} is x. Find the greatest integer less than $100x$.

18. For how many positive integers n less than 1000 is the sum of the digits of the sum of the digits of n equal to 9? (For example, if $n = 47$, the sum of the digits is 11 and the sum of the digits of the sum of the digits is 2).

19. Evaluate the product

$$\prod_{n=1}^{2020}\left(1-\frac{1009}{n}\right)$$

where the $\prod$ symbol represents a product, i.e. $\prod_{i=1}^{n} a_i = a_1 a_2 a_3 \cdots a_n$.

20. Suppose x and y satisfy the equation

$$\log_x y \cdot \log_{y^2}\left(x^3\right) \cdot \log_{x^5}\left(y^8\right) = 3.$$

If $x = 16$, find y.

21. A ball is dropped from 1 foot in the air. Every time it hits the ground, it bounces back 4/5 of the height it fell from. What is the total distance in feet that the ball will travel?

22. Donald drives from his house to the grocery store at a rate of 40 miles per hour down a straight road. It takes him 15 minutes. Returning home, Donald faces terrible traffic and is only able to travel at 8 miles per hour. How long in minutes does it take for Donald to return home?

23. Jerry and Terry each have 4 siblings. Terry's siblings, excluding him, have an average age of 7. Jerry's siblings, excluding him, have an average age of 6. If Jerry and Terry are siblings, find $A_{\text{Jerry}} - A_{\text{Terry}}$, where A_{Jerry} is Jerry's age, and A_{Terry} is Terry's age.

24. Find the smallest integer with 12 positive divisors.

25. The x-coordinate of the point on the x-axis that is equidistant to the points (0,2) and (3,0) can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

26. Albert tells the truth $\frac{3}{5}$ of the time and lies $\frac{2}{5}$ of the time. He randomly picks a marble out of a jar that contains 7 blue marbles and 3 red marble. If he tells you that the marble is red, then the probability that the marble is actually red can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Compute $a + b$.

27. Albert has 2020 balls. He can arrange them in two ways: in a square, or in a triangle shape. In the square format, he places the balls in an a by a square. In the triangle format, he places 1 ball in the first row, 2 in the next row, and n balls in the nth row. If an x by x square is the largest square he can make, then he has x_1 balls left over. If the triangular formation with maximum number of rows has y rows, then there are y_1 balls left. Find $|x_1 - y_1|$.

28. How many factors of $6^4 5^8 8^7$ are perfect squares?

29. If $\sqrt[6]{512} + \sqrt{32} + \sqrt{50} = a^{\frac{1}{b}}$, where a and b are integers and b is minimal, find $\frac{a}{b}$.

30. If $f(x) = \log_{2^x} 2^{x+1}$, find $\prod_{i=1}^{5} f(x)$.

31. Four distinct integers are selected between 0 and 25 inclusive. If the probability that their product is positive is $\frac{a}{b}$, find $a+b$.

32. Circle A is inscribed in equilateral triangle XYZ, which is inscribed in circle B, which is inscribed in square $MNPQ$, which is inscribed in circle C. If circle A has radius 1, and the radius of circle C is denoted by r_c, find r_c^2.

33. A box contains 177 balls, labeled with the numbers 1 through 177. If 5 balls are selected from the box one at a time without replacement, then the probability that each ball's number is greater than all those drawn before it is $\frac{a}{b}$. Find $a+b$.

34. In a regular 5-pointed star with edge length 2020 units, what is the area of the region that contains all points inside the star that are at most 1 unit away from one of the 10 vertices? Express your answer to the nearest whole number.

35. Consider the graph of $y = \sqrt{x^2 + 24}$. How many lattice points are on this graph? (A *lattice point* is a point with integer coordinates. For example, $(2, 3)$ and $(-5, 0)$ are lattice points.)

36. Let a and b be divisors of 2020, chosen independently and uniformly at random (a and b can be the same). The probability that $ab \leq 2020$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

37. Let $P(x) = 8x^3 + 2x^2 + x + 4$. If $P_1(x), P_2(x), \ldots, P_{24}(x)$ are the polynomials that result from permuting the coefficients of $P(x)$, what is the sum of all the possible (not necessarily unique) values of $P(-1)$?

38. Find the number of ordered pairs of positive integers (a, b) such that $\gcd(a, b) + \operatorname{lcm}(a, b) = 61$.

39. Find the number of zeroes at the end of 2048! when it is expressed in base 4.

40. For how many integers x between 0 and 99 inclusive is $\frac{(x!)^3}{(3x)!}$ an integer?

41. Sandy and Randy each have a 100-sided die, labeled 1 to 100. If both of them roll their dice, the probability that Sandy's number is at least twice as large as Randy's number can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Compute $a+b$.

42. Consider the graphs of $x^2 = 25 - y^2$ and $x + 5 = \frac{y^2}{x-5}$. Find the number of points of intersection of the two graphs.

43. Consider the graphs of $|xy| = a$ and $x^2 + y^2 = b^2$. If a and b range over the positive reals, and if $q_1, q_2, \cdots, q_n$ are the number of possible intersections of these two graphs, find $\prod_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i$.

44. A fair 6 sided die is rolled 4 times. The probability that the product of the numbers rolled is prime can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

45. Consider triangles ABC, ABD, and ABE, where A, B, C, D, E are $(0,0)$, $(32,0)$, $(0,255)$, $(0,60)$, $(0,24)$ respectively. If the incenters of ABC, ABD, and ABE are O, P, and Q, find the area enclosed by OPQ.

46. Let a be a positive integer. If

$$\left(\log_{\left(\log_2 a^{\log_2 a^4}\right)} a\right)\left(\log_2 a^{\log_4 a^{\frac{3}{2}}}\right) = 8,$$

what is the value of a?

47. Let the diagonals of cyclic quadrilateral $ABCD$ intersect at K. Suppose $AK = 1, AC = 15, BK = 2$, and $AB = 1.5$. Find $10 \cdot CD$.

48. If $\phi(n)$ is the number of integers $1 \leq p \leq n$ such that p is relatively prime to n, then find the number of even values n between 1 and 100 inclusive for which $\phi(n) = \phi\left(\frac{n}{2}\right)$.

49. Consider a right triangle with side lengths 3, 4, and 5. Let P be the orthocenter of the triangle, let Q be the centroid of the triangle, let R be the circumcenter of the triangle, and let S be the incenter of the triangle. If the area of quadrilateral $PQRS$ is $\frac{a}{b}$ where a and b are relatively prime positive integers, find $a + b$.

50. Let $a_0, a_1, a_2, \ldots$ be an arithmetic sequence of positive integers. If $a_0 + a_1 + \cdots + a_{10} = 209$ and $a_{a_0} + a_{a_1} + \cdots + a_{a_{10}} = 671$, then find a_0.