1. How many cubic centimeters are in 5 cubic meters?

   **Answer: 5000000**

   **Solution:** Because there are 100 centimeters in every meter, there are $10^6 = 1000000$ cm$^3$ in every m$^3$. Thus, our answer is $5 \cdot 1000000 = \boxed{5000000}$.

2. Find the sum of the first 20 positive integers.

   **Answer: 210**

   **Solution:** We use the formula $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. For $n = 20$, this is $\frac{20 \cdot 21}{2} = \boxed{210}$.

3. Find the number of ways to rearrange the letters in $SPEED$ (Note that if the two $E$s switch, then the arrangement is considered the same).

   **Answer: 60**

   **Solution:** There are 5! ways to arrange 5 letters, but we divide by 2 since switching the $E$'s around does not change an arrangement. Thus, there are $\frac{5!}{2} = \boxed{60}$ rearrangements of these letters.

4. Calculate $(\sqrt{17} - 2\sqrt{2})(\sqrt{17} + 2\sqrt{2})$.

   **Answer: 9**

   **Solution:** Using the formula for difference of squares $(a - b)(a + b) = a^2 - b^2$, we have

   $$(\sqrt{17} - 2\sqrt{2})(\sqrt{17} + 2\sqrt{2}) = \sqrt{17}^2 - (2\sqrt{2})^2 = 17 - 8 = \boxed{9}.$$

5. Find the minimum area of a nondegenerate right triangle with integer side lengths.

   **Answer: 6**

   **Solution:** The smallest right triangle with integer side lengths is a $3 - 4 - 5$ triangle, which has area $\frac{3 \cdot 4}{2} = \boxed{6}$.

6. The circumference of a circle is $10\pi$. If the area is $x \cdot \pi$, find $x$.

   **Answer: 25**

   **Solution:** Suppose the circle has radius $r$. Then, the circumference is $2\pi r = 10\pi$, so $r = 5$. Then the area is $\pi r^2 = 25\pi$. Thus, our answer is $\boxed{25}$.

7. Jonah drops a peanut on the floor. It lands at point $A$, bounces 20 centimeters to the right to point $B$, then 15 centimeters down to point $C$, and lastly, 12 centimeters left to point $D$. Find the distance in centimeters between point $A$ and point $D$.

   **Answer: 17**

   **Solution:** The peanut is moved overall 8 centimeters to the right, and 15 centimeters down. Thus, the total distance it traveled is the length of the hypotenuse of a right triangle with leg lengths 8 and 15, which is $\sqrt{8^2 + 15^2} = \boxed{17}$.

8. The triangle $\triangle ABC$ has side lengths $AB = BC = 41$ and $AC = 18$. What is the area of $\triangle ABC$?

   **Answer: 360**

   **Solution:** Splitting the isosceles triangle in half along the altitude to $AC$ yields two right triangles of base 9 and hypotenuse 41. The height is $\sqrt{41^2 - 9^2} = 40$, so we have an area of $\frac{1}{2} \cdot 18 \cdot 40 = \boxed{360}$.

9. Compute the units digit of $333^1 \cdot 333^2 \cdot 333^3 \cdot \dots \cdot 333^{2019} \cdot 333^{2020}$.

   **Answer: 9**

   **Solution:** Note that for any integer $n$, the units digit of a power of $n$ only depends on the units digit of $n$. Thus, combining the exponents, we have to find the units digit of $3^{\frac{2020\cdot 2021}{2}}$. The units digit of $3^n$ cycles every 4 values of $n$, so it suffices to calculate $\frac{2020\cdot 2021}{2} \pmod 4$. This is $\frac{2020\cdot 2021}{2} \equiv 1010 \cdot 2021 \equiv 2 \pmod 4$. Thus, the units digit is $3^2 = \boxed{9}$.

10. How many perfect squares are between 1000 and 10000, inclusive?

    **Answer: 69**

    **Solution:** The lowest such perfect square is $32^2 = 1024$ and the largest such perfect square is $100^2 = 10000$. There are $\boxed{69}$ numbers from 32 to 100 inclusive.

11. The volume of a certain sphere is 32 $\text{m}^3$. Consider a second sphere with radius half as large as the radius of the first sphere. Find the volume of the second sphere.

    **Answer: 4**

    **Solution:** The volume of a sphere is proportional to the radius cubed, so the volume of the second sphere is $(\frac{1}{2})^3 = \frac{1}{8}$ of the first, which is $\frac{32}{8} = \boxed{4}$.

12. What is the sum of all positive integers whose cubes are less than 2020?

    **Answer: 78**

    **Solution:** The smallest positive integer that works is $x = 1$, and the greatest positive integer is $x = 12$, because $12^3 = 1728 < 2000$, while $13^3 = 2197 > 2000$. Thus, our answer is $1 + 2 + \cdots + 12 = \frac{12\cdot 13}{2} = \boxed{78}$.

13. Let $x$ be a two-digit integer. Let $x'$ be the two-digit integer obtained by reversing the digits of $x$. For how many such values of $x$ is the property $|x - x'| = 54$ satisfied? Note that because $x$ and $x'$ are two-digit integers, neither of their tens digits can be 0.

    **Answer: 6**

    **Solution:** Let $x = 10a + b$ and $x' = 10b + a$ for $1 \le a, b \le 9$. Then, if $|x - x'| = 9|a - b| = 54$, we know that $|a - b| = 6$. The possible values of $x$ are thus $17, 28, 39, 71, 82, 93$, for an answer of $\boxed{6}$.

14. Find the sum of all real values of $x$ that satisfy the equation

    $$\left(3x^2 + 2\right)^{\left(4x^2-1\right)} + 2^3 = 3^2.$$

    **Answer: 0**

**Solution:** Rearranging the equation, we have

$$(3x^2+2)^{4x^2-1}=1$$

The only way for this to be true is if $(3x^2+2)=1$ or $-1$, or if $4x^2-1=0$. The former cannot be true because $3x^2$ is always positive, so $3x^2+2\geq 2$. In the second case, $4x^2-1=0$ gives the solutions $x=\pm\frac{1}{2}$. Thus, the sum of the solutions is $\boxed{0}$.

15. Andy starts with a number, takes its square root, multiplies the result by $-4$, adds 2, squares the result, then subtracts 6. If the number he ends with is 30, then what number did he start with?

    **Answer: 4**

    **Solution:** Suppose the starting number is $x$. Then, if we square root $x$, multiply the result by $-4$, add 2, square the result, then subtract 6, we get $\left(-4\sqrt{x}+2\right)^2-6=30$. Adding 6 to both sides and taking a square root yields $|-4\sqrt{x}+2|=6$, which gives us the following cases:

    Case 1. $-4\sqrt{x}+2=6$. This means that $\sqrt{x}=-1$, which is not possible.

    Case 2. $-4\sqrt{x}+2=-6$. In this case, solving for $x$ yields $x=4$.

    Therefore, the number he started with must be $\boxed{4}$.

16. A cube is colored blue on four of its faces, and then cut into $7^3=343$ smaller cubes. If the probability that a randomly selected face of a randomly selected subcube is blue is $\frac{a}{b}$ where $a$ and $b$ are relatively prime positive integers, find $a+b$.

    **Answer: 23**

    **Solution:** There are $4\cdot 7^2$ small blue faces in total, and $6\cdot 7^3$ small faces total. Thus, the probability that a randomly selected face of a randomly selected subcube is blue is $\frac{4\cdot 7^2}{6\cdot 7^3}=\frac{2}{21}$, so the requested sum is $\boxed{23}$.

17. Define $A_0$ to be a square of side length 1. Define $A_1$ as the regular octagon obtained by cutting off a portion of each corner of $A_0$. Define $A_2$ as the regular 16-gon obtained by cutting off a portion of each corner of $A_1$. If this process goes on forever, then the perimeter of $A_{2020}$ is $x$. Find the greatest integer less than $100x$.

    **Answer: 314**

    **Solution:** Notice that as $n$ approaches infinity, $A_n$ approaches a circle of radius $\frac{1}{2}$. As 2020 is sufficiently large, $A_{2020}$ is extremely close to a circle. Since we are only looking for accuracy to the hundredths place, it is safe to use the circle of radius $\frac{1}{2}$ as an approximation. $2*\pi*\frac{1}{2}$ is approximately 3.14. Thus, the requested integer is $\boxed{314}$.

18. For how many positive integers $n$ less than 1000 is the sum of the digits of the sum of the digits of $n$ equal to 9? (For example, if $n=47$, the sum of the digits is 11 and the sum of the digits of the sum of the digits is 2).

    **Answer: 111**

**Solution:** For any positive integers less than 1000, the sum of the digits can be no more than 27. This means that if the number is a multiple of 9, then the sum of the digits will be either 9, 18, or 27, and the sum of the sum of the digits will be 9. There are $\boxed{111}$ positive multiples of 9 less than 1000.

19. Evaluate the product

$$\prod_{n=1}^{2020}\left(1-\frac{1009}{n}\right)$$

where the $\prod$ symbol represents a product, i.e. $\prod_{i=1}^{n} a_i = a_1a_2a_3\cdots a_n$.

**Answer: 0**

**Solution:** Note that the 1009$^{\text{th}}$ term in the product is $1-\frac{1009}{1009}=0$, so the entire product is $\boxed{0}$.

20. Suppose $x$ and $y$ satisfy the equation

$$\log_x y \cdot \log_{y^2}\left(x^3\right)\cdot \log_{x^5}\left(y^8\right)=3.$$

If $x=16$, find $y$.

**Answer: 32**

**Solution:** Simplifying the left hand side,

$$\log_x y \cdot \log_{y^2}\left(x^3\right)\cdot \log_{x^5}\left(y^8\right)=\frac{\log y}{\log x}\cdot\frac{3\log x}{2\log y}\cdot\frac{8\log y}{5\log x}=\frac{12\log y}{5\log x}=3$$

so $\frac{\log y}{\log x}=\log_x y=\frac{5}{4}$. If $x=16$, then $y=x^{\frac{5}{4}}=\boxed{32}$.
Simplifying yields $\frac{12\log_{16}y}{5}=3$, which yields $\log_{16}y=\frac{5}{4}$, giving us that $y=\boxed{32}$.

21. A ball is dropped from 1 foot in the air. Every time it hits the ground, it bounces back 4/5 of the height it fell from. What is the total distance in feet that the ball will travel?

**Answer: 9**

**Solution:** On the first drop and bounce back up, the ball will travel a total of $\frac{9}{5}$ feet. On every subsequent drop and bounce, the ball will travel $\frac{4}{5}$ as far, so the distance traveled each drop and bounce forms a geometric sequence with first term $\frac{9}{5}$ and ratio $\frac{4}{5}$. Using the formula for the sum of a geometric series, the total distance traveled is $\frac{\frac{9}{5}}{1-\frac{4}{5}}=\boxed{9}$.

22. Donald drives from his house to the grocery store at a rate of 40 miles per hour down a straight road. It takes him 15 minutes. Returning home, Donald faces terrible traffic and is only able to travel at 8 miles per hour. How long in minutes does it take for Donald to return home?

**Answer: 75**

**Solution:** Because Donald travels the same distance at $\frac{1}{5}$ the speed, he takes 5 times as long. As the original time was 15 minutes, the new time would be 5 times as long, or $\boxed{75}$ minutes.

23. Jerry and Terry each have 4 siblings. Terry's siblings, excluding him, have an average age of 7. Jerry's siblings, excluding him, have an average age of 6. If Jerry and Terry are siblings, find $A_{\text{Jerry}}-A_{\text{Terry}}$, where $A_{\text{Jerry}}$ is Jerry's age, and $A_{\text{Terry}}$ is Terry's age.

**Answer: 4**

**Solution:** Jerry and Terry each have 4 siblings, so that means that the sum of the ages of all of the siblings is $7 \cdot 4 + A_{\text{Terry}} = 6 \cdot 4 + A_{\text{Jerry}}$. Simplifying, we get $A_{\text{Jerry}} - A_{\text{Terry}} = \boxed{4}$.

24. Find the smallest integer with 12 positive divisors.

**Answer: 60**

**Solution:** For a positive integer $n$ with prime factorization $\prod_{i=1}^{k} p_i^{e_i}$, the number of divisors is $\prod_{i=1}^{k}(e_i + 1)$. If $n$ has 12 positive divisors, then $\prod_{i=1}^{k}(e_i+1) = 1$. To minimize $n$, we want $n$ to have as many prime factors as possible, so it should be in the form $n = p_1^2 p_2 p_3$ for primes $p_1, p_2, p_3$. Taking $p_1 = 2, p_2 = 3$, and $p_3 = 5$, we have $n = \boxed{60}$.

25. The $x$-coordinate of the point on the $x$-axis that is equidistant to the points (0,2) and (3,0) can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

**Answer: 11**

**Solution:** Let this point be $P = (x, 0)$. The distance from P to $(3, 0)$ is $3 - x$, and the distance from $P$ to $(0, 2)$ is $\sqrt{x^2 + 2^2}$. Setting these equal to each other and solving for $x$, we get $x = \frac{5}{6}$. The answer is $5 + 6 = \boxed{11}$.

26. Albert tells the truth $\frac{3}{5}$ of the time and lies $\frac{2}{5}$ of the time. He randomly picks a marble out of a jar that contains 7 blue marbles and 3 red marble. If he tells you that the marble is red, then the probability that the marble is actually red can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime. Compute $a + b$.

**Answer: 32**

**Solution:** If Albert says the marble is red, then either the marble was red and he's telling the truth, or the marble was blue and he was lying. The probability of the first case occurring is $\frac{3}{10} \cdot \frac{3}{5} = \frac{9}{50}$. The probability of the second case occurring is $\frac{7}{10} \cdot \frac{2}{5} = \frac{14}{50}$. However, it is given that one of these cases is true, so the conditional probability that the marble is red is $\frac{\frac{9}{50}}{\frac{9}{50}+\frac{14}{50}} = \frac{9}{23}$. The answer is $9 + 23 = \boxed{32}$.

27. Albert has 2020 balls. He can arrange them in two ways: in a square, or in a triangle shape. In the square format, he places the balls in an $a$ by $a$ square. In the triangle format, he places 1 ball in the first row, 2 in the next row, and $n$ balls in the $n$th row. If an $x$ by $x$ square is the largest square he can make, then he has $x_1$ balls left over. If the triangular formation with maximum number of rows has $y$ rows, then there are $y_1$ balls left. Find $|x_1 - y_1|$.

**Answer: 80**

**Solution:** The largest triangular number less than 2020 is $\frac{63 \cdot 64}{2} = 2016$, so there are 4 balls left over from the largest possible triangular formation. The largest square less than 2020 is $44^2 = 1936$, so there are 84 balls left over from the largest possible square formation. The difference is $84 - 4 = \boxed{80}$.

28. How many factors of $6^4 5^8 8^7$ are perfect squares?

**Answer: 195**

**Solution:** This expression can be simplified to $2^{25}3^4 5^8$. The divisors that are perfect squares are in the form $2^a 3^b 5^c$, where $0 \le a \le 25$, $0 \le b \le 4$, $0 \le c \le 8$; and $a, b, c$ are even. There are $13 \cdot 3 \cdot 5 = \boxed{195}$ such numbers.

29. If $\sqrt[6]{512} + \sqrt{32} + \sqrt{50} = a^{\frac{1}{b}}$, where $a$ and $b$ are integers and $b$ is minimal, find $\frac{a}{b}$.

**Answer: 121**

**Solution:** We have

$$\sqrt[6]{512} + \sqrt{32} + \sqrt{50} = 2\sqrt{2} + 4\sqrt{2} + 5\sqrt{2} = 11\sqrt{2} = 242^{\frac{1}{2}}.$$

Thus, $a = 242$ and $b = 2$, so $\frac{a}{b} = \boxed{121}$.

30. If $f(x) = \log_{2^x} 2^{x+1}$, find $\prod_{i=1}^{5} f(x)$.

**Answer: 6**

**Solution:** Simplifying the expression for $f(x)$,

$$f(x) = \log_{2^x} 2^{x+1} = \frac{(x+1)\log 2}{x \log 2} = \frac{x+1}{x}.$$

Then, $a = \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \boxed{6}$.

31. Four distinct integers are selected between 0 and 25 inclusive. If the probability that their product is positive is $\frac{a}{b}$, find $a + b$.

**Answer: 24**

**Solution:** The product of four nonnegative integers is positive if none of them are zero. There are $\binom{26}{4}$ ways to select four integers out of the set, and $\binom{25}{4}$ to select four *positive* integers out of the set. Thus, the probability that the product of four randomly selected integers will be positive is

$$\frac{\binom{25}{4}}{\binom{26}{4}} = \frac{\frac{26!}{4! \cdot 22!}}{\frac{25!}{4! \cdot 21!}} = \frac{\frac{25 \cdot 24 \cdot 23 \cdot 22}{4!}}{\frac{26 \cdot 25 \cdot 24 \cdot 23}{4!}} = \frac{11}{13}$$

The requested sum is $\boxed{24}$.

32. Circle $A$ is inscribed in equilateral triangle $XYZ$, which is inscribed in circle $B$, which is inscribed in square $MNPQ$, which is inscribed in circle $C$. If circle $A$ has radius 1, and the radius of circle $C$ is denoted by $r_c$, find $r_c^2$.

**Answer: 8**

**Solution:** The length of the line segment from the cetner of the circle to one of the triangles sides, is the radius' length, 1. The radius is the side opposite to the 30 degree angle in a $30 - 60 - 90$ triangle (formed by a angle bisector of the equilateral triangle, also the line that travels through the center of the first circle). Thus, the radius of the second circle is 2. Then, the side length of the square is 4, because it is the same as the diameter of the second circle. Then, the diameter of the last circle is the hypotenuse of a $45-45-90$ triangle, where the leg has length 4, the side length of the square. Thus, the diameter is $4\sqrt{2}$ and the radius is half of that, so $2\sqrt{2}$. Thus, $r_c^2 = \boxed{8}$.

33. A box contains 177 balls, labeled with the numbers 1 through 177. If 5 balls are selected from the box one at a time without replacement, then the probability that each ball's number is greater than all those drawn before it is $\frac{a}{b}$. Find $a+b$.

**Answer: 121**

**Solution:** Notice that it doesn't matter what five numbers are drawn. Out of the 120 ways they can be ordered, there is only one way that keeps them in increasing order, hence our answer of $\frac{1}{120}$, so the requested sum is $\boxed{121}$.

34. In a regular 5-pointed star with edge length 2020 units, what is the area of the region that contains all points inside the star that are at most 1 unit away from one of the 10 vertices? Express your answer to the nearest whole number.

**Answer: 13**

**Solution:** Since a 5-pointed star is a decagon, its interior angles add up to $8 \cdot 180 = 1440°$. The desired region is made of sectors whose angles add up to $1440°$, which is equivalent to 4 full circles with area $4\pi$. Rounded to the nearest whole number, this is $\boxed{13}$.

35. Consider the graph of $y = \sqrt{x^2+24}$. How many lattice points are on this graph? (A *lattice point* is a point with integer coordinates. For example, $(2,3)$ and $(-5,0)$ are lattice points.)

**Answer: 4**

**Solution:** First, note that $y > 0$ for all $x$, since $x^2+24 > 0$. We want to find all pairs of integer solutions $(x,y)$ to $y = \sqrt{x^2+24}$. Squaring both sides and rearranging gives us the equation $(y+x)(y-x) = 24$. Now, $(y+x)$ and $(y-x)$ must both be even if they are integers. We have a few cases to check now: either $y+x$ and $y-x$ are 2 and 12; -2 and 12; 4 and 6; or -4 and -6. Each of these possibilities yields one lattice point, so there are $\boxed{4}$ in total.

36. Let $a$ and $b$ be divisors of 2020, chosen independently and uniformly at random ($a$ and $b$ can be the same). The probability that $ab \leq 2020$ can be written as $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.

**Answer: 37**

**Solution:** We are equally likely to choose $(a,b)$ as $\left(\frac{2020}{a}, \frac{2020}{b}\right)$, and if one of these pairs has product less than 2020, the other has product greater than 2020. So then the probability that $ab < 2020$ is the same as the probability $ab > 2020$, and we can find these values by subtracting the probability that $ab = 2020$ from 1 and dividing by 2. Note that $2020 = 2^2 \cdot 5 \cdot 101$, so it has 12 divisors. Once we choose $a$, there is exactly one choice of $b$ out of 12 possible values that makes $ab = 2020$, so this probability is $\frac{1}{12}$. The probability that $ab < 2020$ is then $\frac{1}{2}\left(1-\frac{1}{12}\right) = \frac{11}{24}$. Thus, the probability that $ab \leq 2020$ is $\frac{1}{12}+\frac{11}{24} = \frac{13}{24}$, and the answer is $13+24 = \boxed{37}$.

37. Let $P(x) = 8x^3+2x^2+x+4$. If $P_1(x), P_2(x), \ldots, P_{24}(x)$ are the polynomials that result from permuting the coefficients of $P(x)$, what is the sum of all the possible (not necessarily unique) values of $P(-1)$?

**Answer: 0**

**Solution:** Each coefficient appears in each position 6 times, so we have $6(15x^3+15x^2+15x+15)$, because 15 is the sum of the coefficients. This is $\boxed{0}$ when $x=-1$.

38. Find the number of ordered pairs of positive integers $(a,b)$ such that $\gcd(a,b)+\operatorname{lcm}(a,b)=61$.

**Answer: 8**

**Solution:** First, $\gcd(a,b)\mid\operatorname{lcm}(a,b)$, so this means $\gcd(a,b)\mid 61$. It cannot be equal to 61 because $\operatorname{lcm}(a,b)>0$, so $\gcd(a,b)=1$ and $\operatorname{lcm}(a,b)=60$.

This means one of $a$ and $b$ contains $2^2$ and the other contains $2^0$, one has $3^1$ and the other $3^0$, and one has $5^1$ and the other $5^0$. There are 2 ways to choose for each of these primes, resulting in $2^3=\boxed{8}$ ways total.

39. Find the number of zeroes at the end of 2048! when it is expressed in base 4.

**Answer: 1023**

**Solution:** We want to find the number of factors of 4 in 2048!, because the number of factors of 4 will be the number of zeroes at the end of 2048! when expressed in base 4. Let's find the number of factors of 2 in 2048!, which we denote by $v_2(2048!)$, and then divide by 2. We have

$$v_2(2048!)=2^{10}+2^9+2^8+2^7+2^6+2^5+2^4+2^3+2^2+2^1+2^0=2^11-1=2047.$$

So we have 1023 factors of 4 and therefore $\boxed{1023}$ zeroes at the end of 1297!.

40. For how many integers $x$ between 0 and 99 inclusive is $\frac{(x!)^3}{(3x)!}$ an integer?

**Answer: 1**

**Solution:** Taking the reciprocal, $\frac{(3x)!}{(x!)^3}$ is the number of ways we can arrange $3x$ letters, with $x$ of the letters being $A$, $x$ of the letters being $B$, and $x$ of the letters being $C$. Because the reciprocal is always a positive integer, the only way for the original expression to be an integer is for both expressions to be 1. The only case for which arranging $3x$ letters, with $x$ of the letters being $A$, $x$ of the letters being $B$, and $x$ of the letters being $C$ has 1 arrangement is when $x=0$. Thus, our answer is $\boxed{1}$. Alternatively, you could notice that for any $x>0$, there will be a prime between $x$ and $3x$, so the denominator have a prime factor and cannot be simplified.

41. Sandy and Randy each have a 100-sided die, labeled 1 to 100. If both of them roll their dice, the probability that Sandy's number is at least twice as large as Randy's number can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime. Compute $a+b$.

**Answer: 5**

**Solution:** We count the total number of cases where Sandy's number is at least twice as large. If Randy rolled a 1, Sandy could have rolled anything from 2 to 100, which is 99 cases. If Randy rolled a 2, Sandy could have rolled 4 to 100, which is 97 cases. This goes on until 1 case when Randy rolls 50. If Randy rolls anything greater than 50, there are 0 cases. The total sum is then the sum of the first 50 odd numbers, which is equal to $50^2 = 2500$. There are $100^2 = 10000$ total possible combinations of the 2 dice, so the probability is $\frac{1}{4}$. The answer is $1 + 4 = \boxed{5}$.

42. Consider the graphs of $x^2 = 25 - y^2$ and $x + 5 = \frac{y^2}{x-5}$. Find the number of points of intersection of the two graphs.

**Answer: 1**

**Solution:** The first equation is $x^2 + y^2 = 25$, which is a circle with radius 5 that is centered at the origin. The next equation is $x^2 - 25 = y^2$ or $x^2 - y^2 = 25$ which is a hyperbola. If we solve $x^2 - y^2 = x^2 + y^2 = 25$ for $x$ and $y$, then we get $y = 0$ and $x = \pm 5$. Thus, our points of intersection are $(5, 0)$ and $(-5, 0)$.

Because we rearranged our equations, we must check that these solutions still fit in the domain and range of the original functions. Our first equation works for $(5, 0)$, because $5^2 = 25 - 0^2 = 25$. Our second equation, however, does not work for $(5, 0)$ because the right hand side divides by $x - 5 = 5 - 5 = 0$. Thus, the point $(5, 0)$ is not a solution. Next, $(-5, 0)$ works for both equations, as $(-5)^2 = 25 - 0^2 = 25$ and $5 - 5 = \frac{0^2}{5+5} = 0$. Thus, the only intersection point is $(-5, 0)$, and there is $\boxed{1}$ intersection.

43. Consider the graphs of $|xy| = a$ and $x^2 + y^2 = b^2$. If $a$ and $b$ range over the positive reals, and if $q_1, q_2, \cdots, q_n$ are the number of possible intersections of these two graphs, find $\prod_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i$.

**Answer: 12**

**Solution:** If the circle reaches farther than the vertices of the hyperbola ($\sqrt{a} < b$), then there are 8 intersections. If the circle reaches exactly to the vertices of the hyperbola ($\sqrt{a} = b$), then there are 4 intersections. If the circle does not reach the vertices of the hyperbola ($\sqrt{a} > b$), then there are 0 intersections. Thus, the requested answer is $0 \cdot 4 \cdot 8 + 0 + 4 + 8 = \boxed{12}$.

44. A fair 6 sided die is rolled 4 times. The probability that the product of the numbers rolled is prime can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

**Answer: 109**

**Solution:** The only way for the product to be prime is if 3 ones and a prime are rolled. There are 4 choices for which die rolls the prime and 3 choices for which prime, so there are 12 combinations of rolls that work. There are $6^4 = 1296$ total possible combinations, so the probability is $\frac{12}{1296} = \frac{1}{108}$. The answer is $1 + 108 = \boxed{109}$.

45. Consider triangles $ABC$, $ABD$, and $ABE$, where $A, B, C, D, E$ are $(0, 0)$, $(32, 0)$, $(0, 255)$, $(0, 60)$, $(0, 24)$ respectively. If the incenters of $ABC$, $ABD$, and $ABE$ are $O$, $P$, and $Q$, find the area enclosed by $OPQ$.

**Answer: 0**

**Solution:** There are a few key observations to make. Each of these triangles are right triangles, and furthermore, the incenters for all of these triangles are along the line $y = x$. This is true because the incircles will have some radii $r$, and these radii form a right angle with the the two legs of the right triangles. Thus, the incenter will be a distance $r$ from the line $y = 0$ and the line $x = 0$. Thus, the point will be at $(r, r)$. Let's define the inradius of $ABC, ABD$, and $ABE$ as $r_1, r_2$, and $r_3$. We need to find the area enclosed in $O(r_1, r_1), P(r_2, r_2)$, and $Q(r_3, r_3)$. However, these points all lie on the line $y = x$ and are thus collinear. Because of this, the area enclosed in $OPQ$ is $\boxed{0}$.

46. Let $a$ be a positive integer. If

$$\left(\log_{\left(\log_2 a^{\log_2 a^4}\right)} a\right)\left(\log_2 a^{\log_4 a^{\frac{3}{2}}}\right) = 8,$$

what is the value of $a$?

**Answer: 16**

**Solution:** Simplifying,

$$(\log_{\log_2 a^{\log_2 a^4}} a)(\log_2 a^{\log_4 a^{1.5}}) = (\log_{4\log_2^2 a} a)(\log_2 a)(\log_4 a^{1.5})$$

$$= \frac{3(\log_{2\log_2 a} a)(\log_2 a)(\log_2 a)}{8} = \frac{3(\log_{2\log_2 a} a)(\log_2^2 a)}{8}.$$

Now, define $x = \log_2 a$, so $a = 2^x$. Our expression is now

$$\frac{3x^2 \log_{2x} 2^x}{8} = \frac{3x^3 \log_{2x} 2}{8} = 8.$$

Thus, $x^3 \log_{2x} 2 = \frac{64}{3}$. The 3 in the denominator implies that we have $\log_{2x} 2 = \log_{2^{3n}} 2$ for some positive integer $n$. If $n = 1$, then $2x = 8$ and $x = 4$. We have $x^3 \log_{2x} 2 = 64 \cdot \log_8 2 = 64 \cdot \frac{1}{3} = \frac{64}{3}$, which is what we wanted! Thus, $x = 4$ and $a = 2^x = \boxed{16}$.

47. Let the diagonals of cyclic quadrilateral $ABCD$ intersect at $K$. Suppose $AK = 1, AC = 15, BK = 2$, and $AB = 1.5$. Find $10 \cdot CD$.

**Answer: 105**

**Solution:** We know that $\triangle AKB \sim \triangle CKD$. We also know that $KC = 14$, so by Power of a Point, $KD = 7$. Therefore, $7AB = CD$, so $CD = 7 \cdot 1.5 = 10.5$. Hence, the answer is $\boxed{105}$.

48. If $\phi(n)$ is the number of integers $1 \le p \le n$ such that $p$ is relatively prime to $n$, then find the number of even values $n$ between 1 and 100 inclusive for which $\phi(n) = \phi\left(\frac{n}{2}\right)$.

**Answer: 25**

**Solution:** Recall that if the prime factorization of a positive integer $n$ is $\prod_{i=1}^k p_i^{e_i}$, then the totient $\phi(n)$ is $\prod_{i=1}^k p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$. We do casework on the number of factors of 2. If $n$ has exactly $d$ factor of 2, then the factor that 2 contributes to the totient is $2^d(1 - \frac{1}{2})$. This is only 1 when $d = 1$, so $\phi(n) = \phi((\frac{n}{2})$ only when $n$ has exactly one factor of 2, or when $n \equiv 2 \pmod 4$. There are $\boxed{25}$ such values.

49. Consider a right triangle with side lengths 3, 4, and 5. Let $P$ be the orthocenter of the triangle, let $Q$ be the centroid of the triangle, let $R$ be the circumcenter of the triangle, and let $S$ be the incenter of the triangle. If the area of quadrilateral $PQRS$ is $\frac{a}{b}$ where $a$ and $b$ are relatively prime positive integers, find $a + b$.

**Answer: 5**

**Solution:** The orthocenter, centroid, circumcenter, and incenter of the triangle are $(0,0)$, $(1, \frac{4}{3})$, $(1.5, 2)$, and $(1,1)$ if the triangles coordinates are $(0,0)$, $(3,0)$, and $(0,4)$. Using the shoelace formula for finding the area of a polygon, we find that the area is $\frac{2+\frac{6}{3}-1.5-2}{2} = 0.25$, which yields $1+4 = \boxed{5}$.

50. Let $a_0, a_1, a_2, \ldots$ be an arithmetic sequence of positive integers. If $a_0 + a_1 + \cdots + a_{10} = 209$ and $a_{a_0} + a_{a_1} + \cdots + a_{a_{10}} = 671$, then find $a_0$.

**Answer: 4**

**Solution:** Let $a_0 = a$ and let the common difference be $d$, so then $a_n = a + nd$. Then we have

$$a_{a_n} = a + a_n \cdot d = a + ad + nd^2,$$

so $a_{a_n}$ is also an arithmetic sequence (with first term $a + ad$ and common difference $d^2$). So then $a_0 + a_1 + \cdots + a_{10} = 11a_5$, and $a_{a_0} + \cdots + a_{a_{10}} = 11a_{a_5}$. So we have $a_5 = \frac{209}{11} = 19$, and then $a_{a_5} = a_{19} = \frac{671}{11} = 61$. So then $d = \frac{61-19}{19-5} = 3$, and $a = 19 - 5d = \boxed{4}$.