FMM 2021 Individual Round

October 23, 2021

1. Let n be the smallest integer greater than 2, such that $\sqrt{n}$, $\sqrt[3]{n}$, $\sqrt[4]{n}$, $\sqrt[5]{n}$ are all integers. If $n = x^y$ for prime x and positive integer y, then compute y. For example, if $n = 81$, then $x = 3$ and $y = 4$.

2. How many regions does the graph of

$$xy(x+y)(y+x)(x+2y)(y+2x) = 0$$

cut the xy-plane into?

3. Let $ABCDEFG$ be a regular heptagon. Given that the measure of the smaller angle of intersection between AD and CF is $\frac{p}{q}$ degrees, where p and q are relatively prime positive integers, find $p+q$.

4. Baron is choosing a team of 1 to 5 people as club officers. He also wants to select a subgroup of the chosen officers of any nonzero size to plan the next club event. There are five people who may be chosen as officers. Find the total number of possible officer-subgroup arrangements.

5. Let r denote the real root of the polynomial $x^3 + 21x + 1$. The infinite sum, $|r - r^4 + r^7 - r^{10} + \cdots|$, can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

6. In the video game *NanoLand*, the player starts at 0 on the number line. The objective is to reach 55 on the number line. There are checkpoints at the points 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55. Every move, the player has a $\frac{1}{2}$ chance of successfully moving forward one unit and a $\frac{1}{2}$ chance of dying and returning to the last checkpoint they touched. If the player dies at a checkpoint, they stay there. What is the expected number of moves needed to reach 55 for the first time?

7. Two circles of radii 1 are separated by a distance of 1 such that they pass through each other's centers. An arclength is chosen uniformly at random between 0 and 2π, and then a random arc with that length is chosen on the circumference of one of the circles. The probability that this arc contains both intersections is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

8. Define $f(n)$ to be the total area enclosed by $\lceil x \rceil \lceil y \rceil = n$. Let n be the number that maximizes $f(n)$ on the interval $n : [1, 900]$. Calculate $n + f(n)$. Note that $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

9. Let ABC be a triangle, and let E and F be the feet of the altitudes from B to AC and C to AB, respectively. If $\cos A = \frac{2}{3}$, $BC = 4$, and the incenter of ABC is on line EF, find the perimeter of ABC.

10. Let the polynomial $P(x) = x^4 + 2x^3 - 5x^2 + 6x + 1$ have roots r_1, r_2, r_3, r_4. Let $Q(x)$ be the monic polynomial of degree 6 with roots $r_i r_j$ for all $1 \le i < j \le 4$. Find $|Q(1)|$.