FMM 2021 Individual Solutions

October 23, 2021

1. Let n be the smallest integer greater than 2, such that $\sqrt{n}$, $\sqrt[3]{n}$, $\sqrt[4]{n}$, $\sqrt[5]{n}$ are all integers. If $n = x^y$ for prime x and positive integer y, then compute y.
Answer: $\boxed{60}$
Solution: To minimize n, we should choose the smallest prime $x = 2$. In order for $\sqrt{n}$, $\sqrt[3]{n}$, $\sqrt[4]{n}$, and $\sqrt[5]{n}$ to all be integers, the exponent y in n must be the least common multiple of 2, 3, 4, and 5. The least common multiple of these numbers is 60, so we may conclude that $n = 2^{60}$ and $y = 60$.

2. How many regions does the graph of
$$xy(x+y)(y+x)(x+2y)(y+2x) = 0$$
cut the xy-plane into?
Answer: $\boxed{10}$
Solution: The graph is made up of 5 lines passing through the origin: $x = 0$, $y = 0$, $x + y = 0$, $x + 2y = 0$, and $2x + y = 0$. Each of the 5 lines cuts the plane along a different axis through the origin, thus creating 10 total regions.

3. Let $ABCDEFG$ be a regular heptagon. Given that the measure of the smaller angle of intersection between AD and CF is $\frac{p}{q}$ degrees, where p and q are relatively prime positive integers, find $p + q$.
Answer: $\boxed{547}$
Solution: Note that segment CF is segment AD rotated about the center of the heptagon by an angle of $\frac{2}{7}(360)$ degrees. Therefore, one of the angles of intersection between the two is also $\frac{2}{7}(360) = \left(\frac{720}{7}\right)$ degrees. Since this is greater than $90°$, the smaller angle of intersection is $180 - \left(\frac{720}{7}\right) = \frac{540}{7}$ degrees, and $p + q = 540 + 7 = 547$.

4. Baron is choosing a team of 1 to 5 people as club officers. He also wants to select a subgroup of the chosen officers of any nonzero size to plan the next club event. There are five people who may be chosen as officers. Find the total number of possible officer-subgroup arrangements.
Answer: $\boxed{211}$
Solution: Select a group of n people from the 5 total to be officers, and multiply this number by the amount of nonempty subsets of a group of size n. In summation notation, this is $\sum_{n=1}^{5} \binom{5}{n}(2^n - 1 = \sum_{n=1}^{5} \binom{5}{n}2^n - \sum_{n=1}^{5} \binom{5}{n}1^n = (2+1)^5 - (1+1)^5 = 211$.

5. Let r denote the real root of the polynomial $x^3 + 21x + 1$. The infinite sum, $|r - r^4 + r^7 - r^{10} + \cdots|$, can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
Answer: $\boxed{22}$
Solution: This is a the sum of an infinite geometric series with first term r and ratio $-r^3$, which we can easily check is between 0 and 1. Therefore, we know the sum is
$$r - r^4 + r^7 - r^{10} + \cdots = \frac{r}{1 + r^3}.$$

However, since we know that $f(r) = r^3 + 21r + 1 = 0$, we have that $r^3 + 1 = -21r$. Thus,

$$r - r^4 + r^7 - r^{10} + \cdots = \frac{r}{r^3+1} = \frac{r}{-21r} = \frac{1}{-21}.$$

The answer is then $1 + 21 = 22$.

6. In the video game *NanoLand*, the player starts at 0 on the number line. The objective is to reach 55 on the number line. There are checkpoints at the points 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55. Every move, the player has a $\frac{1}{2}$ chance of successfully moving forward one unit and a $\frac{1}{2}$ chance of dying and returning to the last checkpoint they touched. If the player dies at a checkpoint, they stay there. What is the expected number of moves needed to reach 55 for the first time?
Answer: $\boxed{4072}$
Solution: We can sum the expected values for each segment of length n from 1 and 10. The expected number of moves it takes to move to the next checkpoint for a segment of length n can be found with states. Labelling $E(k)$ to be the expected number of remaining steps, starting at point k from 0 to n, we have the state equations, $E(n) = 0$, and

$$E(k) = \frac{1}{2}(E(k+1)) + 1) + \frac{1}{2}(E(0) + 1) = \frac{1}{2}E(k+1) + \frac{1}{2}E(0) + 1$$

for each k from 0 to $n-1$. Thus $E(k+1) = 2E(k) - E(0) - 2$. For $k = 0$, we have $E(1) = E(0) - 2$. Then, $E(2) = 2E(1) - E(0) - 2 = E(0) - 2 - 4$. Inducting, if $E(k) = E(0) - 2^{k+1} + 2$, then

$$E(k+1) = 2E(k) - E(0) - 2 = E(0) - 2^{k+2} + 2.$$

Therefore, in general, $E(k) = E(0) - 2^{k+1} + 2$. Plugging in $k = n$, we have $E(n) = 0 = E(0) - 2^{k+1} + 2$, so $E(0) = 2^{k+1} - 2$.

We know that the expected number of moves to cross a segment of length n is $2^{k+1} - 2$. Summing over each segment, the total expected value is then $(2^2 - 2) + (2^3 - 2) + \cdots + (2^{11} - 2) = (2^{12} - 2^1 - 2^0 - 1) - 2(10) = 4072$.

7. Two circles of radii 1 are separated by a distance of 1 such that they pass through each other's centers. An arclength is chosen uniformly at random between 0 and 2π, and then a random arc with that length is chosen on the circumference of one of the circles. The probability that this arc contains both intersections is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
Answer: $\boxed{23}$
Solution: To resolve both the randomness of the location and the length of the arc, consider instead the equivalent process of first choosing 2 points on one of the circles, and then choosing whether the arc formed by these points is major or minor. The only way the arc contains both intersection points is if the two chosen points are both on the same side of the two intersection points, and the arc is oriented correctly. There is a $\frac{2}{3}$ chance a randomly chosen point on one of the circles is on the major arc of the two intersection points and a $\frac{1}{3}$ chance it is on the minor arc between the two circles. Thus there is a

$$\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}$$

chance that both chosen points are on the same side of the two intersection points. There is then a $\frac{1}{2}$ chance that the arc between the two points is oriented the correct way (major or minor) such that the arc does not pass through the two intersection points. Therefore, the desired probability is $\frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$, so the answer is $5 + 18 = 23$.

8. Define $f(n)$ to be the total area enclosed by $\lceil x \rceil \lceil y \rceil = n$. Let n be the number that maximizes $f(n)$ on the interval $n : [1, 900]$. Calculate $n + f(n)$. Note that $\lceil x \rceil$ denotes the smallest integer greater than

or equal to x.
Answer: $\boxed{904}$
Solution: The key observation for this question is that the area enclosed by $\lceil x \rceil \lceil y \rceil = n$ is simply the number of factors of n. We know this because for each integral (x_1, y_1) such that $x_1 y_1 = n$, all x and y such that $x_1 - 1 < x \le x_1$ and $y_1 - 1 < y \le y_1$ are a solution to the equation. Therefore, for each integral factor pair (x_1, y_1), we can create a 1 by 1 square of solutions. Now, we just have to find n that maximizes the number of factor pairs. We know that the number of factors of a number $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$ is $(a_1+1)(a_2+1)\cdots(a_s+1)$. Therefore, we can tell that adding a new prime factor to the number n increases the number of factors by a factor of 2, while adding another prime factor that already exists increases the number of factors by a factor of less than 2. Therefore, we want to maximize the number of different prime factors of n. We find that $2 \cdot 3 \cdot 5 \cdot 7 = 210$ does this. Multiplying by the next prime number 11 would make our number too large. Quickly checking shows that multiplying by 4 would yield the highest number of factors while staying within our limit of $n \le 900$. Therefore, we have $n = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840$ as the n that yields maximal $f(n)$. The number of factors of 840 is $4 \cdot 2 \cdot 2 \cdot 2 = 32$. We have to multiply by 2 to account for the negative factor pairs, yielding $n = 840$ and $f(n) = 64$. Our answer is $840 + 64 = 904$.

9. Let ABC be a triangle, and let E and F be the feet of the altitudes from B to AC and C to AB, respectively. If $\cos A = \frac{2}{3}$, $BC = 4$, and the incenter of ABC is on line EF, find the perimeter of ABC.
Answer: $\boxed{12}$
Solution: Let $BC = a, AC = b, AB = c$, and let the incenter be I. We use area ratios. We have

$$[AIB] = \frac{c}{a+b+c}[ABC],$$

because $[AIB] = \frac{1}{2}ar$ and $[ABC] = \frac{1}{2}(a+b+c)r$ (where r is the inradius). Also, since I is on EF, we have

$$[AIB] = \frac{IF}{EF}[AEB] = \frac{IF}{EF} \cdot \frac{AE}{EC}[ABC].$$

By the Angle Bisector Theorem, $\frac{IF}{IE} = \frac{AF}{AE} = \frac{b}{c}$, since $BCEF$ is cyclic which means $AE \cdot b = AF \cdot c$. So then $\frac{IF}{EF} = \frac{b}{b+c}$. Also, $\frac{AE}{EC} = \frac{c \cos A}{b}$. So then we have

$$[AIB] = \frac{b}{b+c} \cdot \frac{c \cos A}{b} \cdot [ABC],$$

and combining this with the first expression gives $b + c = (a+b+c)\cos A = \frac{2}{3}(a+b+c)$. So then $b + c = 2a = 8$, and the perimeter is 12.

10. Let the polynomial $P(x) = x^4 + 2x^3 - 5x^2 + 6x + 1$ have roots r_1, r_2, r_3, r_4. Let $Q(x)$ be the monic polynomial of degree 6 with roots $r_i r_j$ for all $1 \le i < j \le 4$. Find $|Q(1)|$.
Answer: $\boxed{16}$
Solution: First, we have

$$\begin{aligned} Q(x)^2(x-r_1^2)\cdots(x-r_4^2) &= \prod_{i=1}^{4}(x-r_ir_1)(x-r_ir_2)(x-r_ir_3)(x-r_ir_4) \\ &= \prod_{i=1}^{4} r_i^4 \left(\frac{x}{r_i} - r_1\right)\left(\frac{x}{r_i} - r_2\right)\left(\frac{x}{r_i} - r_3\right)\left(\frac{x}{r_i} - r_4\right) \\ &= \prod_{i=1}^{4} r_i^4 \cdot P\left(\frac{x}{r_i}\right). \end{aligned}$$

So then we can plug in $x = 1$ to find $Q(1)^2$. We have $r_i^4 \cdot P\left(\frac{1}{r_i}\right) = 1 + 2r_i - 5r_i^2 + 6r_i^3 + r_i^4$. Also, since r_i is a root of $P(x)$, we have $r_i^4 - 5r_i^2 + 1 = -2r_i^3 - 6r_i$, so then

$$r_i^4 P\left(\frac{1}{r_i}\right) = -4r_i + 4r_i^3 = -4r_i(1 - r_i^2).$$

So then $Q(1)^2(1 - r_1^2)(1 - r_2^2)(1 - r_3^2)(1 - r_4^2) = (-4)^4 \cdot r_1r_2r_3r_4 \cdot (1 - r_1^2)(1 - r_2^2)(1 - r_3^2)(1 - r_4^2)$. Since none of the r_i are ± 1, then this means $Q(1)^2 = 256r_1r_2r_3r_4 = 256$, since $r_1r_2r_3r_4 = 1$ by Vieta's Formulas. Thus $|Q(1)| = 16$.