FMM 2021 Team Round

October 23, 2021

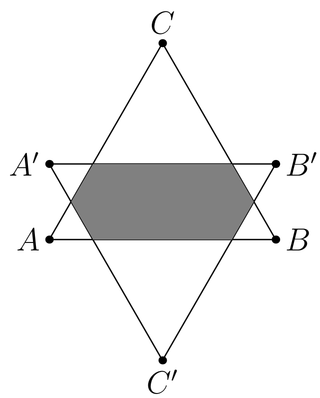
1. Billy scored a 100% on all his math assignments except his most recent one, on which he scored a 75%. He notices that his overall grade in this math class is the same as his friend Angelina's, although her individual assignment grades are different; she scored a 100% on all the assignments except the first one, on which she scored a 65%. If the overall grade is calculated by dividing the total number of points earned by the total number of points possible over all assignments (not all assignments are worth the same number of points), what is the minimum number of points the most recent assignment could have been worth?

2. Suppose x and y are real numbers both greater than 1 that satisfy

$$xy + x + y = 19$$
$$x^2y + y^2x = 84.$$

Find $x^2 + y^2$.

3. Equilateral triangle ABC has vertices A at $(0,0)$, B at $(6,0)$, and C at $(3, 3\sqrt{3})$. Triangle $A'B'C'$ is the reflection of ABC across the line $y = 1$. The area of the region that is in both ABC and $A'B'C'$ can be expressed as $a - b\sqrt{c}$, where c is not divisible by the square of any prime. Compute $a + b + c$.



4. An integer n has a base b representation of 484_b. What is the smallest value of b such that n is the 4th power of an integer?

5. Let $a_k = 10^k + k$ for positive integers k, and call a positive integer representable if it can be written as a sum of some number of distinct a_is. Find the sum of the 15 smallest representable numbers.

6. There are 10 real numbers $a_1, a_2, ..., a_{10}$ such that for all integers k with $2 \leq k \leq 9$,

$$3a_k = 2a_{k-1} + a_{k+1},$$

and $2a_{10} = 2a_9 + a_1$. Then $\frac{a_{10}}{a_1}$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

7. A particle starts on vertex A of square ABCD. Every move, it travels to one of the two adjacent vertices with $\frac{1}{2}$ probability each (it cannot stay on the same vertex or go to the vertex on the opposite corner). Given that the probability that after 12 moves, the particle will have moved to each vertex exactly 3 times (not including its starting position at vertex A) is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

8. Call a positive 10-digit integer n interesting if all the digits of n are distinct, and for all $1 \leq k \leq 10$, if we take n and remove the kth digit from the right, the resulting number is divisible by $2k$. Let N be the largest interesting number. Find $\lfloor \frac{N}{10} \rfloor$.

9. A prison has 100 prisoners numbered prisoner 1 through 100. One day, they form a line in a uniformly random order. Then, a guard walks back and forth between the left and right ends of the line starting at the left end. Each time the guard passes a prisoner, that prisoner leaves the line if his number is between the numbers of the people on his left and right (the first and last prisoners in line never leave). Let the expected number of remaining prisoners in the line after everyone who will eventually leave has left be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10. Let ABC be a triangle with $AB = 13, BC = 14, AC = 15$. Suppose M, N, and P are points inside triangle ABC, such that M is the midpoint of AP, N is the midpoint of BM, and P is the midpoint of CN. Find the area of triangle MNP.

11. Let the score of a number be its distance from the closest power of 2. Let $\frac{m}{n}$ be the sum of the smallest and largest positive real numbers n such that the sum of the scores of n, $2n$, and $3n$ is 128. Compute $m+n$.

12. For integers $n \geq 2$, let $f(n)$ be the product of all odd positive integers less than n. Find the sum of all primes p such that $f(p) \equiv 13 \pmod{p}$.

13. Let ABC be a triangle with centroid G. Suppose there is a point X on the circumcircle of ABC such that $BGCX$ is a parallelogram. If $AB = 12$ and $AC = 16$, then AX can be written as $\frac{m\sqrt{n}}{p}$, where m, n, p are positive integers, m and p are relatively prime, and n is squarefree. Find $m+n+p$.

14. Let f be a randomly chosen function from $\{1, 2, 3, 4, 5, 6\}$ to itself, with each possible function equally likely to be chosen. The expected number of distinct elements in the set

$$\{1, f(1), f(f(1)), f(f(f(1))), \ldots\}$$

can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

15. Let S be the sum of all integers x, with $0 \leq x < 10100$, such that $x^3 \equiv 3^x \pmod{101}$. Find the remainder when S is divided by 10100.