FMM 2021 Team Solutions

October 23, 2021

1. Billy scored a 100% on all his math assignments except his most recent one, on which he scored a 75%. He notices that his overall grade in this math class is the same as his friend Angelina's, although her individual assignment grades are different; she scored a 100% on all the assignments except the first one, on which she scored a 65%. If the overall grade is calculated by dividing the total number of points earned by the total number of points possible over all assignments (not all assignments are worth the same number of points), what is the minimum number of points the most recent assignment could have been worth?
Answer: $\boxed{28}$
Note: $\boxed{7}$ was also accepted as a correct answer due to ambiguity of the problem.
Solution: Notice that they must have both lost the same amount of points since their total grades are the same. Billy's score on the last assignment can be represented as $3a$ out of $4a$, and Angelina's score on the first assignment can be represented as $13b$ out of $20b$ Thus setting the number of points lost for both people to be equal, we have

$$4a - 3a = 20b - 13b,$$

so $a = 7b$. Both a and b must be integers, and the smallest such value for a is when $b = 1$, and so $a = 7$. Therefore, the minimum number of points the most recent assignment could have been worth is $4a = 28$.

2. Suppose x and y are real numbers both greater than 1 that satisfy

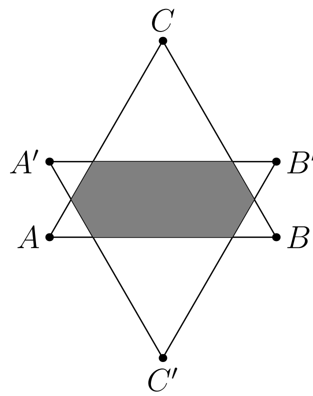
$$xy + x + y = 19$$
$$x^2y + y^2x = 84.$$

Find $x^2 + y^2$.
Answer: $\boxed{25}$
Solution: Let $a = x + y$ and $b = xy$. From here we get

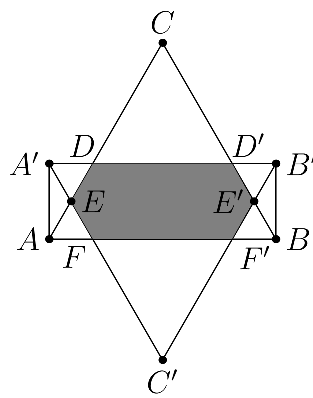
$$a + b = 19$$
$$ab = 84$$

It follows that $a = 7$ and $b = 12$, or $a = 12$ and $b = 7$. Solving for the latter case of $x + y = 12$ and $xy = 7$ yields a solution with either x or y needing to be less than 1. Solving for the first case gives either $x = 3$ and $y = 4$ or $x = 4$ and $yb = 3$. In either case, $x^2 + y^2 = 25$.

3. Equilateral triangle ABC has vertices A at $(0, 0)$, B at $(6, 0)$, and C at $(3, 3\sqrt{3})$. Triangle $A'B'C'$ is the reflection of ABC across the line $y = 1$. The area of the region that is in both ABC and $A'B'C'$ can be expressed as $a - b\sqrt{c}$, where c is not divisible by the square of any prime. Compute $a + b + c$.



Answer: $\boxed{17}$
Solution:



Let D, D', E, E', F, and F' represent the intersection points between the triangles as labelled in the figure. Consider rectangle $AA'B'B$. The length AB is 6 and the length AA' is 2 because A' is at $(0, 2)$. area is $6 \cdot 2 = 12$. Now we need to find and subtract the areas of DEA', $A'EA$, AEF, $D'E'B'$, $B'E'B$, and $BE'F'$. An important observation is that each of these triangles has equal area as each is made of two congruent 30-60-90 triangles.

We look for the area of AEF. The altitude from E to AF is 1 because it is half the length of AA'. By using a 30-60-90 triangle, we can get that the length of AE is $\frac{2}{\sqrt{3}}$. Since AEF is equilateral, the area is

$$\left(\frac{\sqrt{3}}{4}\right)\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{3}.$$

Since there are 6 triangles with this area, their total area is $6 \cdot \frac{\sqrt{3}}{3} = 2\sqrt{3}$. We subtract this from the area of $AA'B'B$ to get that the shaded region has area $12 - 2\sqrt{3}$. The answer is thus $12 + 2 + 3 = 17$.

4. An integer n has a base b representation of 484_b. What is the smallest value of b such that n is the 4th power of an integer?
Answer: $\boxed{17}$
Solution: We write n in base 10 as

$$4b^2 + 8b + 4 = (2b + 2)^2.$$

For this to be a perfect 4th power, $2b + 2$ must be a perfect square. Since $b > 8$, we have

$$2b + 2 > 18.$$

The first perfect square greater than 18 is 25. This doesn't work because $2b + 2 = 25$ doesn't have an integer solution. The next perfect square is 36. Setting $2b + 2 = 36$, we get that $b = 17$ works.

5. Let $a_k = 10^k + k$ for positive integers k, and call a positive integer representable if it can be written as a sum of some number of distinct a_is. Find the sum of the 15 smallest representable numbers.
Answer: $\boxed{88960}$
Solution: Note that $a_k > a_{k-1} + a_{k-2} + \ldots + a_1$ for all k, so then the representable numbers in increasing order are $a_1, a_2, a_2 + a_1, a_3, a_3 + a_1, a_3 + a_2, a_3 + a_2 + a_1, \ldots$. So then the first 15 representable numbers are the 15 sums using some (nonempty) subset of a_1, a_2, a_3, a_4. Each of a_1, a_2, a_3, a_4 is in exactly 8 of these subsets, so the sum is

$$8(a_1 + a_2 + a_3 + a_4) = 8(11 + 102 + 1003 + 10004) = 8(11120) = 88960.$$

6. There are 10 real numbers $a_1, a_2, ..., a_{10}$ such that for all integers k with $2 \leq k \leq 9$,

$$3a_k = 2a_{k-1} + a_{k+1},$$

and $2a_{10} = 2a_9 + a_1$. Then $\frac{a_{10}}{a_1}$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
Answer: $\boxed{1535}$
Solution: Rearranging terms in the given equations, we have

$$2a_k - 2a_{k-1} = a_{k+1} - a_k$$

for each $2 \leq k \leq 9$. This means that the difference between consecutive terms doubles for every increment of k.

Let $d = a_2 - a_1$. We have

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_1 + d + 2d \\ a_4 &= a_1 + d + 2d + 4d \\ &\vdots \\ a_9 &= a_1 + 255d \\ a_{10} &= a_1 + 511d \end{aligned}$$

Substituting the last two equations into $2a_{10} = 2a_9 + a_1$ gives

$$2(a_1 + 511d) = 2(a_1 + 255d) + a_1$$

$$512d = a_1.$$

We can now compute,

$$\frac{a_{10}}{a_1} = \frac{512d + 511d}{512d} = \frac{1023}{512},$$

so $p + q = 1023 + 512 = 1535$.

7. A particle starts on vertex A of square ABCD. Every move, it travels to one of the two adjacent vertices with $\frac{1}{2}$ probability each (it cannot stay on the same vertex or go to the vertex on the opposite corner). Let $\frac{m}{n}$ be the probability that after 12 moves, the particle will have moved to each vertex exactly 3 times (not including its starting position at vertex A), where m and n are relatively prime positive integers. Find $m + n$.
Answer: $\boxed{281}$
Solution: The key observation here is that after an odd number of moves, the particle can only be

at vertex B or D and after an even number of moves, the particle can only be at vertex A or C. Furthermore, the particle is equally likely to be at B or D on any odd numbered move, since it is $\frac{1}{2}$ probability each regardless of whether it was at A or C before. The same goes for even numbered moves. Therefore, we can consider the odd numbered moves as a string of B's and D's and the even numbered moves as a string of A's and C's. There are $\binom{6}{3} = 20$ ways to have 3 B's and 3 D's in the list out of a total of $2^6 = 64$ possible strings. So the probability of having 3 B's and 3 D's is $\frac{20}{64} = \frac{5}{16}$. Similarly, the probability of having 3 A's and 3 C's is also $\frac{5}{16}$. Thus, the probability of having 3 A's, B's, C's, and D's is $(\frac{5}{16})^2 = \frac{25}{256}$, and the answer is $5 + 256 = 281$.

8. Call a positive 10-digit integer n interesting if all the digits of n are distinct, and for all $1 \leq k \leq 10$, if we take n and remove the kth digit from the right, the resulting number is divisible by $2k$. Let N be the largest interesting number. Find $\lfloor \frac{N}{10} \rfloor$.
Answer: $\boxed{897531264}$
Solution: Let $n = \overline{a_{10}a_9a_8\ldots a_1}$. First, from $k = 10$ we know that $a_1 = 0$. Then, a number is divisible by 3 or 9 if and only if its sum of digits is. Since $0 + 1 + \ldots + 9$ is a multiple of 9, we must have that $a_9 = 9$ and that a_3, a_6 are 3 and 6 in some order.

From $k = 2$ we know $\overline{a_3 0}$ is a multiple of 4, so a_3 is even, which means $a_3 = 6$. From $k = 4$ we know $\overline{a_3 a_2 0}$ is a multiple of 8, so a_2 is a multiple of 4 (since a_3 is even), meaning it is 4 or 8.

Since we want n to be large, take $a_{10} = 8$. Then our number is $n = \overline{89a_8a_73a_5a_4640}$. From $k = 8$, $\overline{a_4640}$ must be a multiple of 16, so a_4 is even, which means $a_4 = 2$. Now we have $n = \overline{89a_8a_73a_52640}$, and we have already satisfied all conditions except for $k = 7$. For $k = 7$, we need $\overline{89a_83a_52640}$ to be a multiple of 7, where a_5, a_8 are $1, 5$, or 7. Since $10, 10^2, 10^3$ are $3, 2, -1 \pmod 7$, we can calculate

$$\overline{89a_83a_52640} \equiv 10(4 + 6 \cdot 3 + 2 \cdot 2 - a_5 - 3 \cdot 3 - 2 \cdot a_8 + 9 + 3 \cdot 8) \equiv 10(1 - a_5 - 2a_8) \pmod 7.$$

Therefore, $a_8 = 7$ and $a_5 = 1$, so then $a_7 = 5$ and $N = 8975312640$. Thus, our answer is $\lfloor \frac{8975312640}{10} \rfloor = 897531264$.

9. A prison has 100 prisoners numbered prisoner 1 through 100. One day, they form a line in a uniformly random order. Then, a guard walks back and forth between the left and right ends of the line starting at the left end. Each time the guard passes a prisoner, that prisoner leaves the line if his number is between the numbers of the people on his left and right (the first and last prisoners in line never leave). Let the expected number of remaining prisoners in the line after everyone who will eventually leave has left be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
Answer: $\boxed{205}$
Solution: We observe that the guard need only make 1 pass from the left to the right end of the line before everyone's position in line is fixed. We can now use linearity of expectation to figure out the average number of prisoners that leave on this first pass. Each of the 98 prisoners that are in between the two ends has a $\frac{1}{3}$ chance of having a number between the numbers on the two prisoners to his left and right, so a $\frac{1}{3}$ chance of leaving the line. Therefore, the expected number of remaining prisoners in the line is

$$2 + 98\left(1 - \frac{1}{3}\right) = \frac{202}{3},$$

and $m + n = 202 + 3 = 205$.

10. Let ABC be a triangle with $AB = 13, BC = 14, AC = 15$. Suppose M, N, and P are points inside triangle ABC, such that M is the midpoint of AP, N is the midpoint of BM, and P is the midpoint of CN. Find the area of triangle MNP.
Answer: $\boxed{12}$
Solution: Let $[MNP] = S$. Then $[CMP] = S$ as well, since P is the midpoint of CN, so then $[APC] = 2S$, since M is the midpoint of AP. Similarly, $[AMB] = [BNC] = 2S$, so then $[ABC] = 7S$.

Since $[ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$ by Heron's Formula, $S = \frac{1}{7} \cdot 84 = 12$.

11. Let the score of a number be its distance from the closest power of 2. Let $\frac{m}{n}$ be the sum of the smallest and largest positive real numbers n such that the sum of the scores of n, $2n$, and $3n$ is 128. Compute $m + n$.
Answer: $\boxed{803}$
Solution: Let the score of n be $S(n)$. First note that $S(2n) = 2S(n)$, so

$$g(n) = S(n) + 2S(n) + 3S(n) = 3S(n) + S(3n).$$

Let 2^k be the power of two such that n is between 2^k and 2^{k+1}. There are three cases we can consider: n is between 2^k and $\frac{4(2^k)}{3}$, between $\frac{4(2^k)}{3}$ and $\frac{3(2^k)}{2}$, or between $\frac{3(2^k)}{2}$ and 2^{k+1}.

In the first case, $S(n) = n - 2^k$ and $S(3n) = 4(2^k) - 3n$, so $g(n) = 2^k$.

In the second, $S(n) = n - 2^k$ and $S(3n) = 3n - 4(2^k)$, so $g(n) = 6n - 7(2^k)$.

In the final case, $S(n) = 2^{k+1} - n$ and $S(3n) = 3n - 4(2^k)$, so $g(n) = 2^{k+1}$

We notice through the above casework that the only plausible values of n such that $g(n) = 128$ are between 64 and 256. Notice also that in the first and last case, the value of $g(n)$ only depends on 2^k, and is a power of 2 over the entire domain of the case.

Therefore, we see that the minimum value of n such that $g(n) = 128$ is when n is between 64 and 128, and occurs at the boundary between the second and third cases, when

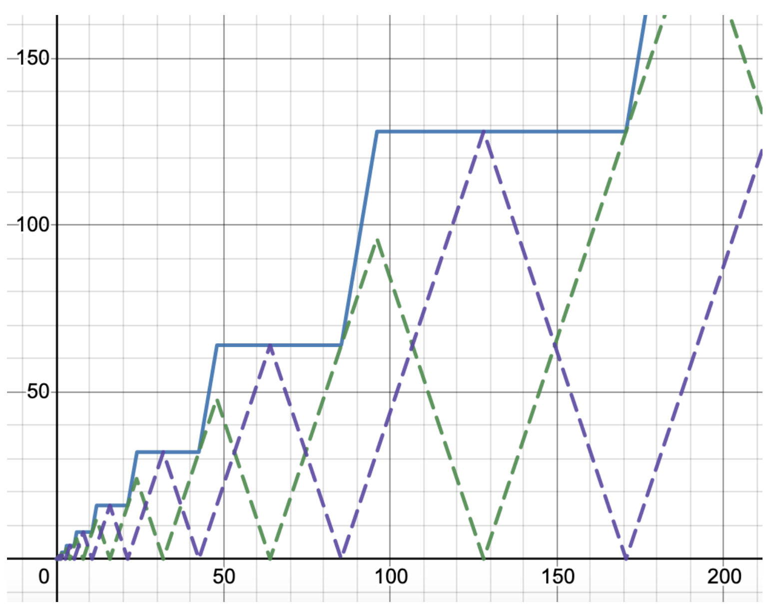
$$n = \frac{3(2^k)}{2} = \frac{3(64)}{2} = 96,$$

and the maximum value of n such that $g(n) = 128$ is when n is between 128 and 256, and occurs at the boundary between the first and second cases, when

$$n = \frac{4(2^k)}{3} = \frac{4(128)}{3} = \frac{512}{3}.$$

This gives us a sum of $96 + \frac{512}{3} = \frac{800}{3}$, so $m + n = 800 + 3 = 803$.

A graph of the functions is shown below. The green function is $3S(n)$, the purple function is $S(3n)$, and the blue function is their sum, $g(n)$.



12. For integers $n \geq 2$, let $f(n)$ be the product of all odd positive integers less than n. Find the sum of all primes p such that $f(p) \equiv 13 \pmod{p}$.
Answer: $\boxed{27}$
Solution: First, $p = 2$ works, so now assume p is an odd prime. Then

$$f(p) = 1 \cdot 3 \cdot 5 \cdots (p-2) \equiv (-1)^{\frac{p-1}{2}} \cdot (p-1) \cdot (p-3) \cdots 4 \cdot 2 \pmod{p},$$

by replacing every term k with $p - k$. This means

$$f(p)^2 \equiv (-1)^{\frac{p-1}{2}} (p-1)! \equiv (-1)^{\frac{p+1}{2}} \pmod{p},$$

since $(p-1)! \equiv -1 \pmod{p}$ by Wilson's Theorem. Since $f(p) \equiv 13 \pmod{p}$, this means p divides $13^2 - 1 = 168$ or $13^2 + 1 = 170$, so then $p = 3, 5, 7,$ or 17. We can check that $3, 5,$ and 17 work and 7 does not work, so then the answer is $2 + 3 + 5 + 17 = \boxed{27}$.

13. Let ABC be a triangle with centroid G. Suppose there is a point X on the circumcircle of ABC such that $BGCX$ is a parallelogram. If $AB = 12$ and $AC = 16$, then AX can be written as $\frac{m\sqrt{n}}{p}$, where m, n, p are positive integers, m and p are relatively prime, and n is squarefree. Find $m + n + p$.
Answer: $\boxed{29}$
Solution: Let BC, AC, AB be a, b, c. Let M be the midpoint of BC. Since the diagonals of a parallelogram bisect each other, this means GX passes through M, so A, G, M, X are collinear. Additionally, $MX = GM = \frac{1}{3}AM$. So then by Power of a Point on M, we have $BM \cdot MC = AM \cdot MX$, so then

$$\frac{1}{4}a^2 = AM \cdot \frac{1}{3}AM = \frac{1}{3}AM^2.$$

By Stewart's Theorem we can calculate $a(AM^2 + \frac{1}{4}a^2) = \frac{1}{2}ab^2 + \frac{1}{2}ac^2$, so then

$$AM^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2.$$

This means

$$\frac{1}{4}a^2 = \frac{1}{3} \cdot \left(\frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2\right),$$

so then $a^2 = \frac{b^2+c^2}{2}$ and $AM^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{8}(b^2 + c^2) = \frac{3}{8}(b^2 + c^2) = \frac{3}{8} \cdot (12^2 + 16^2) = 150$. So $AM = 5\sqrt{6}$, and

$$AX = AM + MX = \frac{4}{3}AM = \frac{20\sqrt{6}}{3},$$

so the answer is $20 + 6 + 3 = 29$.

14. Let f be a randomly chosen function from $\{1, 2, 3, 4, 5, 6\}$ to itself, with each possible function equally likely to be chosen. The expected number of distinct elements in the set

$$\{1, f(1), f(f(1)), f(f(f(1))), \ldots\}$$

can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
Answer: $\boxed{1223}$
Solution: Once we get a repeated value in the list $\{1, f(1), f(f(1)), \ldots\}$, all values after it will be repeated. Therefore, the number of distinct values in the set is the number of values we get before the first repeat. If P_k is the probability that the first k elements have no repeated values, then the expected value of the number of distinct values is $P_1 + P_2 + P_3 + \ldots + P_6$. This is because the number of distinct values is k with probability $P_k - P_{k+1}$, so the expected value is $1(P_1 - P_2) + 2(P_2 - P_3) + \ldots + 5(P_5 - P_6) + 6P_6 = P_1 + \ldots + P_6$.

Clearly $P_1 = 1$. Then, P_2 is the probability that $f(1) \neq 1$, which is $\frac{5}{6}$. For P_3, we need to have $f(1)$ not being 1 and $f(f(1))$ being neither 1 nor $f(1)$, which has probability $\frac{5}{6} \cdot \frac{4}{6}$. Continuing this way, we get $P_k = \frac{5}{6} \cdot \frac{4}{6} \cdots \frac{(7-k)}{6}$, because at each step we need to go to a value distinct from all we have been to already. Thus the expected value is $1+\frac{5}{6}+\frac{5}{6}\cdot\frac{4}{6}+\ldots+\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\cdot\frac{2}{6}\cdot\frac{1}{6} = \frac{899}{324}$, and the answer is $899+324 = 1223$.

15. Let S be the sum of all integers x, with $0 \leq x < 10100$, such that $x^3 \equiv 3^x \pmod{101}$. Find the remainder when S is divided by 10100.
Answer: $\boxed{5050}$
Solution: First, $x \not\equiv 0 \pmod{101}$. Raising both sides to the 67th power (because $67 \cdot 3 \equiv 1 \pmod{100}$),

$$x^{201} \equiv x \equiv 3^{67x} \pmod{101}$$

by Fermat's Little Theorem. Letting $a = 3^{67}$, we have $x \equiv a^x \pmod{101}$.

The value of $a^x \pmod{101}$ only depends on $x \pmod{100}$ since $a^{100} \equiv 1 \pmod{101}$. For each of the choices of $x \pmod{100}$, there is exactly 1 choice of $x \pmod{101}$ satisfying the equation. So by the Chinese Remainder Theorem, each choice of $x \pmod{100}$ gives exactly one choice of $x \pmod{100 \cdot 101}$.

Thus,

$$S \equiv 0 + 1 + 2 + \cdots + 99 \equiv 50 \pmod{100},$$

and

$$S \equiv a^0 + a^1 + \cdots + a^{99} \equiv \frac{a^{100} - 1}{a - 1} \equiv 0 \pmod{101}.$$

Combining these, $S \equiv 5050 \pmod{10100}$.